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Paul Doukhan, Gabriel Lang, Donatas Surgailis

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RANDOMLY FRACTIONALLY INTEGRATED PROCESSES¹

P. Doukhan

LS-CREST and University Cergy Pontoise, 95011 Cergy-Pontoise Cedex, France

G. Lang

Laboratoire GRESE, ENGREF, 19 av. du Maine, 75732 Paris Cedex 15, France

D. Surgailis

Institute of Mathematics and Informatics, Akademijos 4, LT-08663 Vilnius, Lithuania
(e-mail: sdonatas@ktl.mii.lt)

Abstract. Philippe *et al.* [9], [10] introduced two distinct time-varying mutually invertible fractionally integrated filters $A(\mathbf{d})$, $B(\mathbf{d})$ depending on an arbitrary sequence $\mathbf{d} = (d_t)_{t \in \mathbb{Z}}$ of real numbers; if the parameter sequence is constant $d_t \equiv d$, then both filters $A(\mathbf{d})$ and $B(\mathbf{d})$ reduce to the usual fractional integration operator $(1 - L)^{-d}$. They also studied partial sums limits of filtered white noise nonstationary processes $A(\mathbf{d})\varepsilon_t$ and $B(\mathbf{d})\varepsilon_t$ for certain classes of deterministic sequences \mathbf{d} . The present paper discusses the randomly fractionally integrated stationary processes $X_t^A = A(\mathbf{d})\varepsilon_t$ and $X_t^B = B(\mathbf{d})\varepsilon_t$ by assuming that $\mathbf{d} = (d_t, t \in \mathbb{Z})$ is a random iid sequence, independent of the noise (ε_t) . In the case where the mean $\bar{d} = \mathbb{E}d_0 \in (0, 1/2)$, we show that large sample properties of X^A and X^B are similar to FARIMA(0, \bar{d} , 0) process; in particular, their partial sums converge to a fractional Brownian motion with parameter $\bar{d} + (1/2)$. The most technical part of the paper is the study and characterization of limit distributions of partial sums for nonlinear functions $h(X_t^A)$ of a randomly fractionally integrated process X_t^A with Gaussian noise. We prove that the limit distribution of those sums is determined by a conditional Hermite rank of h . For the special case of a constant deterministic sequence d_t , this reduces to the standard Hermite rank used in Dobrushin and Major [2].

Keywords: first keyword, second keyword, third keyword, very long keyword, very long keyword, very long keyword.

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1. INTRODUCTION

Philippe *et al.* [9] introduced time-varying fractional filters $A(\mathbf{d})$, $B(\mathbf{d})$ defined by

$$A(\mathbf{d})x_t = \sum_{j=0}^{\infty} a_j(t)x_{t-j}, \quad B(\mathbf{d})x_t = \sum_{j=0}^{\infty} b_j(t)x_{t-j}, \quad (1.1)$$

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where $\mathbf{d} = (d_t, t \in \mathbb{Z})$ is a given function of $t \in \mathbb{Z}$, and $a_0(t) = b_0(t) := 1$,

$$a_j(t) := \left(\frac{d_{t-1}}{1}\right)\left(\frac{d_{t-2}+1}{2}\right)\left(\frac{d_{t-3}+2}{3}\right)\cdots\left(\frac{d_{t-j}+j-1}{j}\right), \quad (1.2)$$

$$b_j(t) := \left(\frac{d_{t-1}}{1}\right)\left(\frac{d_{t-j}+1}{2}\right)\left(\frac{d_{t-j+1}+2}{3}\right)\cdots\left(\frac{d_{t-2}+j-1}{j}\right), \quad j \geq 1. \quad (1.3)$$

If $d_t \equiv d$ is a constant, then $A(\mathbf{d}) = B(\mathbf{d}) = (I - L)^{-d}$ is the usual fractional integration operator ($Lx_t := x_{t-1}$ is the backward shift). Let $a_j^-(t), b_j^-(t)$ be defined as in (1.2), (1.3), with $\mathbf{d} = (d_t)_{t \in \mathbb{Z}}$ replaced by $-\mathbf{d} = (-d_t)_{t \in \mathbb{Z}}$. For arbitrary sequence $\mathbf{d} = (d_t)_{t \in \mathbb{Z}}$ such that $d_t \notin \mathbb{Z}$ ($\forall t \in \mathbb{Z}$), the coefficients in (1.2), (1.3) satisfy the orthogonality relation

$$\sum_{j=0}^n b_j^-(t) a_{n-j}(t-j) = \sum_{j=0}^n a_j^-(t) b_{n-j}(t-j) = \delta_n.$$

In other words, the filters $A(\mathbf{d}), B(-\mathbf{d})$ are mutually inverse: $A(\mathbf{d})^{-1} = B(-\mathbf{d})$, $B(\mathbf{d})^{-1} = A(-\mathbf{d})$ (see Philippe *et al.* [9], [10]). The above mentioned papers studied long memory behavior and partial sums limits of nonstationary processes $X_t^A = A(\mathbf{d})\varepsilon_t$, $X_t^B = B(\mathbf{d})\varepsilon_t$, where (ε_t) is a white noise, for certain classes of deterministic sequences \mathbf{d} admitting (possibly different) Cesaro limits d_+, d_- at $+\infty, -\infty$, respectively, and showed that the limit behavior of partial sums of X_t^A and X_t^B essentially depends on the limits d_+, d_- alone.

The present paper studies long-memory properties of the *randomly fractionally integrated* processes

$$X_t^A := \sum_{j=0}^{\infty} a_j(t) \varepsilon_{t-j}, \quad X_t^B := \sum_{j=0}^{\infty} b_j(t) \varepsilon_{t-j}, \quad (1.4)$$

where $(\varepsilon_t, t \in \mathbb{Z})$ is an iid white noise, $a_j(t), b_j(t)$ are given in (1.2)–(1.3), and $\mathbf{d} = (d_t, t \in \mathbb{Z})$ is an iid sequence with mean $\bar{d} = \mathbb{E}d_0 \in (0, 1/2)$ and finite variance (the sequences $(\varepsilon_t, t \in \mathbb{Z})$ and $\mathbf{d} = (d_t, t \in \mathbb{Z})$ are assumed mutually independent). Then $(X_t^A), (X_t^B)$ in (1.4) are well-defined, strictly stationary, and ergodic processes. We show in Section 1 that long-memory properties of X^A and X^B are very similar to FARIMA(0, \bar{d} , 0); in particular, partial sums of X^A and X^B converge to a fractional Brownian motion with Hurst parameter $H = \bar{d} + (1/2) \in (1/2, 1)$.

The main objective of the present paper is to study long-memory properties of *non-linear* functionals $(h(X_t^A))_{t \in \mathbb{Z}}, (h(X_t^B))_{t \in \mathbb{Z}}$ of stationary processes in (1.4). There is a large literature concerning limit theorems for instantaneous nonlinear functionals of Gaussian and linear processes with long memory, see, e.g., Dobrushin and Major [2], Taqqu [15], Ho and Hsing [6], Surgailis [13], and the references therein. Consider a linear process $Y_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}$ with slowly decaying coefficients $a_j \sim c_0 j^{d-1}$

($c_0 \neq 0$, $d \in (0, 1/2)$ and iid innovations $(\varepsilon_t) \sim \text{iid}(0, 1)$). It was shown in these papers that the limit distribution of suitably normalized partial sums processes $\sum_{t=1}^{[N\tau]} h(Y_t)$ is determined by the *Appell rank* of the nonlinear function h , or the integer

$$k_* := \min \left\{ k \geq 1: \frac{d^k \mathbb{E}h(Y_0 + x)}{dx^k} \Big|_{x=0} \neq 0 \right\}, \quad (1.5)$$

under some additional moment and regularity conditions on h and ε_0 .

Note that, for Gaussian processes (Y_t) (with $Y_0 \sim N(0, 1)$), the Appell rank in (1.5) coincides with the Hermite rank, or the smallest $k \geq 1$ with $c_k \neq 0$ in the Hermite expansion $h(x) = \sum_{k=0}^{\infty} c_k H_k(x)/k!$. Under the long memory condition $k_*(2d - 1) < 1$, the limit of partial sums of $h(Y_t)$ is a so-called Hermite process of order k_* (see Section 2 for definition).

To study the asymptotic behavior of nonlinear functionals of “random FARIMA” processes in (1.4), we assume that the iid sequence (d_t) is bounded and the iid sequence (ε_t) is Gaussian; moreover, our discussion is limited to the process (X_t^A) . Extensions to more general (d_t) , (ε_t) , and the filter (X_t^B) are possible but not easy. The assumption of conditional gaussianity allows us to use conditional Hermite expansions and simplifies estimation of remainder terms. Our main result, Theorem 3.1, states that the limit distribution of partial sums $\sum_{t=1}^{[N\tau]} h(X_t^A)$ is determined by the integer $k_*^A := \min\{k \geq 1: \beta_k \neq 0\}$, where β_k ’s are defined via conditional Hermite expansion of h ; it turns out that, under certain regularity conditions, the integer k_*^A can also be defined as

$$k_*^A = \min \left\{ k \geq 1: \frac{d^k \mathbb{E}h(X_0^A + x Q(0))}{dx^k} \Big|_{x=0} \neq 0 \right\}, \quad (1.6)$$

where $Q(0) := \lim_{j \rightarrow \infty} a_j(0)/\psi_j(\bar{d})$, and $\sum_{j=0}^{\infty} \psi_j(\bar{d})L^j = (1 - L)^{-\bar{d}}$ is FARIMA(0, \bar{d} , 0) filter. While k_* in (1.5) and k_*^A in (1.6) look similarly, the former quantity is expressed via the marginal distribution of (Y_t) at $t = 0$ alone, and the latter involves the joint distribution $(X_0^A, Q(0))$; moreover, the derivative in (1.6) does not seem to be related to any polynomial expansion of h .

2. LINEAR FUNCTIONALS

Everywhere below, $(\varepsilon_t, t \in \mathbb{Z}) \sim \text{iid}(0, 1)$ is a standard iid sequence of rv’s with zero mean and unit variance, and $\mathbf{d} = (d_t, t \in \mathbb{Z})$ is another sequence of iid rv’s, with mean $\bar{d} = \mathbb{E}d_0$ and variance $\sigma^2 := \mathbb{E}(d_0 - \mathbb{E}d_0)^2 < \infty$; the sequences $(\varepsilon_t)_{t \in \mathbb{Z}}$ and $\mathbf{d} = (d_t)_{t \in \mathbb{Z}}$ are assumed mutually independent. Let $\delta_t := d_t - \mathbb{E}d_t$ denote the centered iid rv’s. We also assume that $d_t \notin \mathbb{Z}_- := \{0, -1, -2, \dots\}$ a.s.

Let $\psi_j(d)$, $j \geq 0$, be the FARIMA(0, d , 0) coefficients defined by $(1 - z)^{-d} = \sum_{j=0}^{\infty} \psi_j(d)z^j$. Recall that, for $0 < d < 1/2$, the autocovariance of the process $Y_t := (1 - L)^{-d} \varepsilon_t$ decays as t^{2d-1} , more precisely,

$$\mathbb{E}Y_0 Y_t = \sum_{j=0}^{\infty} \psi_j(d) \psi_{t+j}(d) = \frac{\Gamma(1 - 2d)}{\Gamma(d)\Gamma(1 - d)} t^{2d-1} (1 + O(t^{-1})) \quad (2.1)$$

(see Hosking [7] and Kokoszka and Taqqu [8]). Introduce the Hermite process of order $k = 1, 2, \dots$:

$$J_k(\tau) := \frac{1}{\Gamma(d)^k} \int_{\mathbb{R}^k} \left\{ \int_0^\tau \prod_{i=1}^k (t - u_i)_+^{d-1} dt \right\} W(du_1) \dots W(du_k), \quad (2.2)$$

given by a k -tuple Itô-Wiener integral with respect to a standard Gaussian white noise $W(ds)$ with zero mean and variance ds ; $u_+^{d-1} := u^{d-1}$ for $u > 0$, $:= 0$ otherwise. The process J_k in (2.2) is well defined for $1 \leq k < 1/(1-2d)$ and is self-similar with index $H = 1 - (1-2\bar{d})k/2$. The process J_1 is a fractional Brownian motion (up to the constant $\mathbb{E}^{1/2} J_1^2(1)$). Other properties of J_k in (2.2) including the expressions for $\mathbb{E} J_k^2(1)$ can be found in Taqqu [15].

THEOREM 2.1. (i) *Let $\bar{d} < 1/2$. Then the series X_t^A, X_t^B in (1.4) converge a.s. and in L^2 for all $t \in \mathbb{Z}$ and define strictly stationary and ergodic processes with zero mean $\mathbb{E} X_t^A = \mathbb{E} X_t^B = 0$ and respective covariances*

$$\mathbb{E} X_0^A X_t^A = \sum_{j=0}^{\infty} \mathbb{E} a_j a_{t+j}, \quad \mathbb{E} X_0^B X_t^B = \sum_{j=0}^{\infty} \mathbb{E} b_j b_{t+j}.$$

(ii) *Let $0 < \bar{d} < 1/2$. Then*

$$\mathbb{E} X_0^A X_t^A = \mathbb{E} Y_0 Y_t (1 + O(t^{-1} \log t)), \quad \mathbb{E} X_0^B X_t^B = c_B^2 \mathbb{E} Y_0 Y_t (1 + O(t^{-\bar{d}})) \quad (2.3)$$

as $t \rightarrow \infty$, where $\mathbb{E} Y_0 Y_t$ is the autocovariance of FARIMA(0, \bar{d} , 0) (see (2.1)), and the constant c_B^2 is given in (2.8) below. Moreover,

$$N^{-\bar{d}-(1/2)} \sum_{t=1}^{[N\tau]} X_t^A \rightarrow_{D[0,1]} c(\bar{d}) J_1(\tau), \quad (2.4)$$

$$N^{-\bar{d}-(1/2)} \sum_{t=1}^{[N\tau]} X_t^B \rightarrow_{D[0,1]} c(\bar{d}) c_B J_1(\tau), \quad (2.5)$$

where $(J_1(\tau), \tau \geq 0)$ is a fractional Brownian motion with (Hurst) parameter $H = (1/2) + \bar{d}$ (see (2.2)).

Proof. (i) Assume first that $\bar{d} \notin \mathbb{Z}_-$. Define

$$Q_A(s, t) := \frac{a_{t-s}(t)}{\psi_{t-s}(\bar{d})} = \prod_{s \leq u < t} \left(1 + \frac{\delta_u}{\bar{d} + t - u - 1} \right) \quad (s < t), \quad (2.6)$$

$$Q_B(s, t) := \frac{\bar{d} b_{t-s}(t)}{d_{t-1} \psi_{t-s}(\bar{d})} = \prod_{s \leq u < t-1} \left(1 + \frac{\delta_u}{\bar{d} + u - s + 1} \right) \quad (s < t-1), \quad (2.7)$$

$Q_B(s, s+1) := 1$. The expectations

$$\begin{aligned} \mathbb{E} Q_A^2(s, t) &= \prod_{s \leq u < t} \left(1 + \frac{\sigma^2}{(\bar{d} + t - u - 1)^2} \right) \leq c_A^2, \\ \mathbb{E} Q_B^2(s, t) &= \prod_{s \leq u < t-1} \left(1 + \frac{\sigma^2}{(\bar{d} + u - s + 1)^2} \right) \leq c_B^2 \end{aligned}$$

are uniformly bounded in $s < t$ by finite constants

$$c_A^2 := \prod_{i \geq 0} \left(1 + \frac{\sigma^2}{(\bar{d} + i)^2} \right), \quad c_B^2 := \prod_{i \geq 1} \left(1 + \frac{\sigma^2}{(\bar{d} + i)^2} \right), \quad (2.8)$$

respectively. Therefore,

$$\begin{aligned} \mathbb{E} a_{t-s}^2(t) &= \psi_{t-s}^2(\bar{d}) \mathbb{E} Q_A^2(s, t) \leq c_A^2 \psi^2(\bar{d}), \\ \mathbb{E} b_{t-s}^2(t) &= \psi_{t-s}^2(\bar{d}) \left(\frac{\mathbb{E} d_{t-1}^2}{\bar{d}^2} \right) \mathbb{E} Q_B^2(s, t) \leq c_B^2 \frac{\bar{d}^2 + \sigma^2}{\bar{d}^2} \psi_{t-s}^2(\bar{d}), \end{aligned}$$

implying $\sum_{j=0}^{\infty} \mathbb{E} a_j^2(t) < \infty$ and $\sum_{j=0}^{\infty} \mathbb{E} b_j^2(t) < \infty$ by the well-known property of FARIMA coefficients and, thus, the convergences of the series in (1.4). The stationarity and ergodicity properties of these series are easy. This proves part (i) for $\bar{d} \notin \mathbb{Z}_-$. In the case $\bar{d} \in \mathbb{Z}_-$, the above argument requires minor modifications.

(ii) Consider the covariance of X^A . Using $\mathbb{E} Q_A(s, t) = 1$, similarly as in the proof of (i), one has

$$\mathbb{E} X_0^A X_t^A = \sum_{s \leq 0} \psi_{-s}(\bar{d}) \psi_{t-s}(\bar{d}) \mathbb{E} Q_A(s, 0) Q_A(s, t) = \mathbb{E} Y_0 Y_t + R_t^A,$$

where

$$R_t^A := \sum_{j=t+1}^{\infty} \psi_j(\bar{d}) \psi_{t+j}(\bar{d}) \Lambda_j^A(t), \quad \Lambda_j^A(t) := \prod_{i=1}^j \left(1 + \frac{\sigma^2}{(\bar{d} + i - 1)(\bar{d} + t + i - 1)} \right) - 1.$$

Note that $\sup_{j > t} |\Lambda_j^A(t)| = O(t^{-1} \log t)$; indeed,

$$1 \leq 1 + \Lambda_j^A(t) \leq \exp \left\{ \sum_{i=1}^j \log \left(1 + \frac{\sigma^2}{(\bar{d} + i - 1)(\bar{d} + t + i - 1)} \right) \right\}$$

$$\begin{aligned} &\leq \exp \left\{ \sum_{i=1}^j \frac{\sigma^2}{(\bar{d} + i - 1)(\bar{d} + t + i - 1)} \right\} \\ &\leq \exp \left\{ C \sum_{i=1}^{\infty} \frac{\sigma^2}{i(t+i)} \right\} = \exp \{ O(\log t/t) \} = 1 + O\left(\frac{\log t}{t}\right). \end{aligned}$$

Then $R_t^A = O(t^{2\bar{d}-2} \log t)$ (see (2.1)), proving the first asymptotic in (2.3). Next,

$$\begin{aligned} \mathbb{E} X_0^B X_t^B &= \sum_{s \leq -1} \psi_{-s}(\bar{d}) \psi_{t-s}(\bar{d}) \mathbb{E} Q_B(s, 0) Q_B(s, t) \frac{d_{-1} d_{t-1}}{\bar{d}^2} + \psi_0(\bar{d}) \psi_t(\bar{d}) \\ &= c_B^2 \mathbb{E} Y_0 Y_t + R_t^B, \end{aligned}$$

where $R_t^B := \sum_{j=0}^{\infty} \psi_j(\bar{d}) \psi_{t+j}(\bar{d}) \Lambda_j^B$ and

$$\Lambda_j^B := \prod_{k=1}^{j-1} \left(1 + \frac{\sigma^2}{(\bar{d} + k)^2} \right) \left\{ \left(1 + \frac{\sigma^2}{\bar{d}(\bar{d} + j)} \right) \prod_{p=j}^{\infty} \left(1 + \frac{\sigma^2}{(\bar{d} + p)^2} \right) - 1 \right\} \quad (j \geq 1),$$

$\Lambda_0^B := 1 - c_B^2$, satisfy $\Lambda_j^B = O(j^{-1})$. From this $R_t^B = O(t^{\bar{d}-1})$ easily follows, proving the second asymptotic in (2.3).

To show (2.4), decompose $X_t^A = Y_t + Z_t^A$, where

$$Z_t^A := \sum_{j=0}^{\infty} \psi_j(\bar{d}) (Q_A(t-j, t) - \mathbb{E} Q_A(t-j, t)) \varepsilon_{t-j}$$

is a short-memory process satisfying $\sum_{t=1}^N Z_t^A = O_{\mathbb{P}}(N^{1/2})$, which follows by evaluating the covariance $\text{cov}(Q_A(s, 0), Q_A(s, t))$ ($s \leq 0 \leq t$) similarly as above. Then the convergence of finite-dimensional distributions in (2.4) is immediate from the corresponding convergence of FARIMA process (Y_t) , and the tightness follows by the Kolmogorov criterion using the fact about the covariance of (X_t^A) in (2.3). The proof of the tightness part in (2.5) is completely analogous; however, the convergence of finite-dimensional distributions is a little more complicated. Namely, one represents X_t^B as $X_t^B = Y_t^B + Z_t^B$, where $Y_t^B = \sum_{s \leq t} \psi_{t-s}(\bar{d}) \varepsilon_s^B$ is FARIMA(0, \bar{d} , 0) process in strictly stationary *backward martingale difference* innovations (ε_s^B) defined by

$$\varepsilon_s^B := \varepsilon_s Q_B(s, \infty) = \varepsilon_s \prod_{j \geq 1} \left(1 + \frac{\delta_{s+j-1}}{\bar{d} + j} \right),$$

with variance $\mathbb{E}(\varepsilon_s^B)^2 = c_B^2$ given in (2.8). The fact that finite-dimensional distributions of partial sums of (Y_t^B) tend to those of the limit process in (2.5) can be easily proved

using the scheme of discrete stochastic integrals as in Philippe *et al.* [9], [10]. The “remainder term” Z_t^B in the above decomposition of X_t^B is given by

$$Z_t^B := (\varepsilon_t^B - \varepsilon_t) + \frac{\delta_t}{\bar{d}} \sum_{s < t} \psi_{t-s}(\bar{d}) Q_B(s, t) \varepsilon_s + \sum_{s < t} \psi_{t-s}(\bar{d}) (Q_B(s, t) - Q_B(s, \infty)) \varepsilon_s.$$

The proof of the relation $\sum_{t=1}^N Z_t^B = O_{\mathbb{P}}(N^{1/2})$ follows using similar argument as in the proof of (2.3), and we omit the details for the sake of brevity. This completes the proof of Theorem 2.1.

3. NONLINEAR FUNCTIONALS

In this section, we study long-memory properties of *nonlinear* processes $(h(X_t^A))_{t \in \mathbb{Z}}$, where h is a real function such that $\mathbb{E}h^2(X_0^A) < \infty$. As noted in Introduction, the discussion is limited to the process (X_t^A) in (1.4), so we omit the superscript “A” in the following notation, i.e., we write $X_t \equiv X_t^A$, $Q(s, t) \equiv Q_A(s, t)$, etc. Because of the difficulty of dealing with nonlinear functionals, the assumptions on (d_t) and (ε_t) now are strengthened as follows.

Assumption 1. The sequence (ε_t) is iid $N(0, 1)$ -distributed.

Assumption 2. The sequence (d_t) is iid, independent of (ε_t) , with mean $\bar{d} \in (0, 1/2)$ and finite variance $\sigma^2 = \text{var}(d_0) < \infty$; moreover, there is a constant $D < \infty$ such that

$$|\delta_t| \leq D \quad a.s. \quad (3.1)$$

We use the following notation: $\psi_j := \psi_j(\bar{d})$,

$$Q(t) := Q(-\infty, t) = \prod_{i=1}^{\infty} \left(1 + \frac{\delta_{t-i}}{\bar{d} + i - 1}\right), \quad (3.2)$$

$$A^2(t) := \sum_{j=0}^{\infty} a_j^2(t) = \sum_{j=0}^{\infty} Q^2(t-j, t) \psi_j^2 = \sum_{j=0}^{\infty} \psi_j^2 \prod_{i=1}^j \left(1 + \frac{\delta_{t-i}}{\bar{d} + i - 1}\right)^2. \quad (3.3)$$

Also, let $\mathcal{D}_t := \sigma\{d_s : s \leq t\}$ and $\mathcal{D} := \bigvee_t \mathcal{D}_t$ denote the sigma-algebras generated by the iid sequence (d_t) . From Assumptions 1 and 2 it follows that the process (X_t) in (1.4) is a *conditionally Gaussian process* given the sigma-algebra \mathcal{D} , with zero conditional mean and the conditional variance $A^2(t)$, i.e.,

$$\mathbb{E}[X_t | \mathcal{D}] = 0, \quad \mathbb{E}[X_t^2 | \mathcal{D}] = A^2(t).$$

For any $A > 0$ and any real function $h(x)$, $x \in \mathbb{R}$, with $\mathbb{E}h^2(X) < \infty$ ($X \sim N(0, A^2)$), we can write the Hermite expansion

$$h(x) = \sum_{k=0}^{\infty} \frac{g_k(A)}{k!} H_k(x; A), \quad (3.4)$$

where

$$\begin{aligned} g_k(A) &:= A^{-2k} \mathbb{E}[h(X) H_k(X; A)] \\ &= \frac{1}{\sqrt{2\pi} A^{1+2k}} \int_{\mathbb{R}} h(x) H_k(x; A) e^{-x^2/2A^2} dx, \end{aligned} \quad (3.5)$$

and $H_k(x; A) := A^k H_k(x/A)$, $k \geq 0$, are Hermite polynomials with standard deviation $A > 0$; $H_k(x)$, $k \geq 0$, are standard Hermite polynomials with generating function $\sum_{k=0}^{\infty} z^k H_k(x)/k! = e^{zx - z^2/2}$. Finally,

$$\beta_k := \mathbb{E}[g_k(A(0)) Q^k(0)], \quad k = 0, 1, \dots, \quad (3.6)$$

where $g_k(\cdot)$, $A(0)$, and $Q(0)$ are defined in (3.5), (3.3), and (3.2), respectively.

THEOREM 3.1. *Let Assumptions 1 and 2 be fulfilled. Let $h: \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function such that*

$$\mathbb{E}|h(BX_0)|^a < \infty \quad (3.7)$$

for some $B > 1$ and $a > 2$. Let k_^A be the smallest integer $k \geq 1$ such that $\beta_k \neq 0$:*

$$k_*^A := \min \{k \geq 1: \beta_k \neq 0\}, \quad (3.8)$$

and let $(1 - 2\bar{d})k_^A < 1$. Then*

$$N^{(1-2\bar{d})(k_*^A/2)-1} \sum_{t=1}^{[N\tau]} (h(X_t) - \mathbb{E}h(X_t)) \rightarrow_{D[0,1]} \frac{\beta_{k_*^A}}{k_*^A!} J_{k_*^A}(\tau), \quad (3.9)$$

where $J_k(\tau)$ is a k th order Hermite process given in (2.2) with $d = \bar{d}$.

Remark 3.1. We show in the proof of Theorem 3.1 below that the coefficients β_k (3.6) are well defined for any $k \geq 1$. Moreover, as noted in Introduction, under additional conditions on the function $h(\cdot)$, these coefficients can be identified with the derivatives in (1.6), i.e.,

$$\beta_k = \frac{d^k \mathbb{E}h(X_0 + wQ(0))}{dw^k} \Big|_{w=0}. \quad (3.10)$$

Indeed, $X_0 =_{\text{law}} A(0)Z$, where $Z \sim N(0, 1)$ does not depend on $(A(0), Q(0)) \equiv (A, Q)$. Then, assuming that the differentiations and integrations by parts below are legitim, we can rewrite the right-hand side of (3.10) as

$$\left(\int_{\mathbb{R}_+ \times \mathbb{R}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(Az + wQ) e^{-z^2/2} dz \mathbb{P}(dA, dQ) \right)^{(k)} \Big|_{w=0}$$

$$\begin{aligned}
&= \int_{\mathbb{R}_+ \times \mathbb{R}} \frac{Q^k}{\sqrt{2\pi}} \int_{\mathbb{R}} h^{(k)}(Az) e^{-z^2/2} dz \mathbb{P}(dA, dQ) \\
&= \int_{\mathbb{R}_+ \times \mathbb{R}} \frac{Q^k}{\sqrt{2\pi} A^k} \int_{\mathbb{R}} h(Az) H_k(z) e^{-z^2/2} dz \mathbb{P}(dA, dQ) = \beta_k;
\end{aligned}$$

see definitions (3.6), (3.5).

Remark 3.2. Condition $\mathbb{E}|h(BX_0)|^a < \infty$ ($B > 1$) entails $\mathbb{E}|h(X_0)|^a < \infty$. Indeed,

$$\begin{aligned}
\mathbb{E}|h(BX_0)|^a &= \mathbb{E} \frac{1}{\sqrt{2\pi} A(0)} \int |h(BA(0)x)|^a e^{-x^2/2A^2(0)} dx \\
&= \mathbb{E} \frac{1}{\sqrt{2\pi} BA(0)} \int |h(A(0)x)|^a e^{-x^2/2B^2A^2(0)} dx \\
&\geq B^{-1} \mathbb{E} \frac{1}{\sqrt{2\pi} A(0)} \int |h(A(0)x)|^a e^{-x^2/2A^2(0)} dx = B^{-1} \mathbb{E}|h(X_0)|^a
\end{aligned}$$

or $\mathbb{E}|h(X_0)|^a \leq B \mathbb{E}|h(BX_0)|^a < \infty$.

The proof of Theorem 3.1 follows technical lemmas discussed in the following section. In these lemmas, Assumptions 1 and 2 are imposed without explicit reference to them. On the other hand, some of these statements hold under weaker conditions without the assumption of gaussianity of (ε_t) or boundedness of (d_t) .

4. SOME TECHNICAL LEMMAS

Recall from (2.6) the definition $Q(t-j, t) \equiv Q_j(t)$, i.e.,

$$Q_j(t) = a_j(t)/\psi_j = \prod_{i=1}^j \left(1 + \frac{\delta_{t-i}}{\bar{d} + i - 1}\right) \quad (j \geq 1), \quad Q_0(t) := 1.$$

LEMMA 4.1. *For any $t \in \mathbb{Z}$, $Q_j(t) \rightarrow Q(t)$ ($j \rightarrow \infty$) a.s. Moreover, for any $p \geq 2$, there exists a constant $C = C_p$ such that, for any $t \in \mathbb{Z}$ and $j \geq 0$,*

$$\mathbb{E}|Q_j(t)|^p \leq C, \quad \mathbb{E}A^{2p}(t) \leq C \quad (4.1)$$

and such that

$$\mathbb{E}|Q_j(t) - Q(t)|^p \leq Cj^{-p/2}. \quad (4.2)$$

Proof. Denote

$$\xi_j := Q_{j+1}(t) - Q_j(t) = Q_j(t) \frac{\delta_{t-j-1}}{\bar{d} + j}.$$

Note that the random variables $(\xi_j)_{j \geq 1}$ are orthogonal (in fact, they are martingale differences) and, therefore,

$$Q_{j+1}(t) = 1 + \sum_{i=0}^j \xi_i \quad (4.3)$$

is the sum of orthogonal rv's. According to a result of Stout [11], the series (4.3) converges a.s., provided that

$$\sum_{j=1}^{\infty} \log^2(j) \mathbb{E} \xi_j^2 < \infty. \quad (4.4)$$

In our case,

$$\mathbb{E} \xi_j^2 = \frac{\sigma^2}{(\bar{d} + j)^2} \mathbb{E} Q_j^2(t), \quad (4.5)$$

where

$$\mathbb{E} Q_j^2(t) = \prod_{i=1}^j \mathbb{E} \left(1 + \frac{\delta_{t-i}}{\bar{d} + i - 1} \right)^2 = \prod_{i=1}^j \left(1 + \frac{\sigma^2}{(\bar{d} + i - 1)^2} \right) < C \quad (4.6)$$

is bounded. Therefore, (4.4) holds implying the first part of the lemma.

Let us prove the first bound in (4.1). We shall need the following general inequality: for any $p \geq 2$, $0 < \delta < 1$, and any rv δ with $\mathbb{E} \delta = 0$, $\mathbb{E} |\delta|^p < \infty$, there exists a constant $C = C_p$ such that

$$\mathbb{E} |1 + a\delta|^p \leq 1 + Ca^2. \quad (4.7)$$

Indeed, write $\mathbb{E} |1 + a\delta|^p = \sum_{k=1}^4 J_k$, where $J_1 := \mathbb{E} |1 + a\delta|^p I(a\delta \leq -1)$, $J_2 := \mathbb{E} (1 + a\delta)^p I(-1 < a\delta \leq -1/2)$, $J_3 := \mathbb{E} (1 + a\delta)^p I(|a\delta| < 1/2)$, and $J_4 := \mathbb{E} (1 + a\delta)^p I(a\delta > 1/2)$. Then $J_1 \leq \mathbb{E} |a\delta|^p \leq Ca^2$, $J_2 \leq \mathbb{P}(|a\delta| > 1/2) \leq 4a^2 \mathbb{E} \delta^2 \leq Ca^2$, $J_4 \leq C \mathbb{E} |a\delta|^p \leq Ca^2$, and

$$J_3 = 1 + pa \mathbb{E} \delta I(|a\delta| < 1/2) + O(a^2) = 1 - pa \mathbb{E} \delta I(|a\delta| \geq 1/2) + O(a^2) = 1 + O(a^2),$$

since $\mathbb{E} |\delta| I(|a\delta| \geq 1/2) \leq Ca \mathbb{E} \delta^2 \leq Ca^2$. This proves (4.7).

Applying (4.7) with $a = \frac{1}{\bar{d} + i - 1}$, $\delta = \delta_{t-i}$, we obtain

$$\mathbb{E} |Q_j(t)|^p = \prod_{i=1}^j \mathbb{E} \left| 1 + \frac{\delta_{t-i}}{\bar{d} + i - 1} \right|^p \leq \prod_{i=1}^j \left(1 + \frac{C}{i^2} \right) \leq C$$

and, thus, prove the first bound in (4.1).

Consider the second bound in (4.1). Since $A^2(t) = \sum_{j=0}^{\infty} Q_j^2(t) \psi_j^2$, the Minkowski inequality and the previous bound together yield

$$(\mathbb{E}[A^{2p}(t)])^{1/p} \leq \sum_{j=0}^{\infty} \psi_j^2 (\mathbb{E}[Q_j^{2p}(t)])^{1/p} \leq C \sum_{j=0}^{\infty} \psi_j^2 < C.$$

It remains to prove (4.2). In the case $p = 2$, we immediately get

$$\mathbb{E}(Q(t) - Q_j(t))^2 = \mathbb{E}\left(\sum_{i=j}^{\infty} \xi_i\right)^2 = \sum_{i=j}^{\infty} \mathbb{E}\xi_i^2 \leq C \sum_{i=j}^{\infty} j^{-2} \leq Cj^{-1}$$

due to (3.5) and (3.6).

Let $p > 2$. Since (ξ_j) is a martingale difference sequence, by the B rkholder inequality we have

$$\mathbb{E}|Q(t) - Q_j(t)|^p = \mathbb{E}\left|\sum_{i=j}^{\infty} \xi_i\right|^p \leq C_p \mathbb{E}\left(\sum_{i=j}^{\infty} \xi_i^2\right)^{p/2}.$$

By the H lder inequality, we have

$$\begin{aligned} \sum_{i=j}^{\infty} \xi_i^2 &= \sum_{i=j}^{\infty} (i + \bar{d})^{-1} (i + \bar{d})^{-1} \delta_{t-i-1}^2 Q_i^2(t) \\ &\leq \left(\sum_{i=j}^{\infty} (i + \bar{d})^{-p/(p-2)}\right)^{(p-2)/p} \left(\sum_{i=j}^{\infty} (i + \bar{d})^{-p/2} |\delta_{t-i-1}|^p |Q_i(t)|^p\right)^{2/p} \\ &\leq Cj^{-2/p} \left(\sum_{i=j}^{\infty} (i + \bar{d})^{-p/2} |\delta_{t-i-1}|^p |Q_i(t)|^p\right)^{2/p}. \end{aligned}$$

Therefore, using (4.1), we get

$$\begin{aligned} \mathbb{E}|Q(t) - Q_j(t)|^p &\leq Cj^{-1} \sum_{i=j}^{\infty} (i + \bar{d})^{-p/2} \mathbb{E}|\delta_{t-i-1}|^p |Q_i(t)|^p \\ &\leq Cj^{-1} \sum_{i=j}^{\infty} (i + \bar{d})^{-p/2} \leq Cj^{-p/2}, \end{aligned}$$

proving the lemma.

LEMMA 4.2. Let $p_i, k_i \geq 0$, $q_i \in \mathbb{Z}$, $i = 1, 2$, be given integers, and let

$$\phi_i(t) := A^{q_i}(t) Q_{j_i}^{p_i}(t) M_i(A(t)), \quad (4.8)$$

where

$$M_i(A) := \int_{\mathbb{R}} h(x) x^{k_i} e^{-x^2/2A^2} dx. \quad (4.9)$$

Then there exists a constant $C = C(k_i, p_i, q_i, i = 1, 2) < \infty$ such that, for any $t \in \mathbb{Z}$, $j_1, j_2 \geq 0$,

$$|\text{cov}(\phi_1(t), \phi_2(0))| \leq Ct^{-1} \log t.$$

Proof. Write the telescoping expansion

$$\phi_i(t) - \mathbb{E}\phi_i(t) = \sum_{j=1}^{\infty} U_i(t, j), \quad (4.10)$$

where

$$\begin{aligned} U_i(t, j) &:= \mathbb{E}[\phi_i(t) | \mathcal{D}_{t-j}] - \mathbb{E}[\phi_i(t) | \mathcal{D}_{t-j-1}] \\ &= \int_{\mathbb{R}} h(x) x^{k_i} dx \left(\mathbb{E}[A^{q_i}(t) Q^{p_i}(t) e^{-x^2/2A^2(t)} | \mathcal{D}_{t-j}] \right. \\ &\quad \left. - \mathbb{E}[A^{q_i}(t) Q^{p_i}(t) e^{-x^2/2A^2(t)} | \mathcal{D}_{t-j-1}] \right). \end{aligned}$$

By orthogonality,

$$\begin{aligned} |\text{cov}(\phi_1(t), \phi_2(0))| &= \left| \sum_{j=1}^{\infty} \mathbb{E}[U_1(t, t+j) U_2(0, j)] \right| \\ &\leq \sum_{j=1}^{\infty} (\mathbb{E}U_1^2(t, t+j))^{1/2} (\mathbb{E}U_2^2(0, j))^{1/2}. \end{aligned}$$

Thus, the lemma follows from the bound

$$\mathbb{E}U_i^2(t, j) \leq Cj^{-2}. \quad (4.11)$$

Fix t, j, i ; then with $k = k_i$, $p = p_i$, $q = q_i$, $U(t, j) = U_i(t, j)$, by definition we have

$$U(t, j) = \mathbb{E}[V(t, j) | \mathcal{D}_{t-j}], \quad (4.12)$$

where

$$V(t, j) := \int_{\mathbb{R}} h(x) x^k dx \left(A^q(t) Q^p(t) e^{-x^2/2A^2(t)} - \mathbb{E}[A^q(t) Q^p(t) e^{-x^2/2A^2(t)} \mid \mathcal{F}_{t-j}] \right) \quad (4.13)$$

and $\mathcal{F}_s = \sigma\{\delta_u: u \neq s\}$. With $\delta := \delta_{t-j}$, write also

$$\begin{aligned} A^2(t) &= \sum_{k=0}^{j-1} \psi_k^2 \prod_{i=1}^k \left(1 + \frac{\delta_{t-i}}{\bar{d} + i - 1} \right)^2 \\ &\quad + \left(1 + \frac{\delta}{\bar{d} + j - 1} \right)^2 \sum_{k=j}^{\infty} \psi_k^2 \prod_{i=1, i \neq j}^k \left(1 + \frac{\delta_{t-i}}{\bar{d} + i - 1} \right)^2 \\ &=: \alpha_1^2 + \left(1 + \frac{\delta}{\bar{d} + j - 1} \right)^2 \alpha_2^2, \end{aligned} \quad (4.14)$$

$$\tilde{A}^2(t) := A^2(t)|_{\delta=0} = \alpha_1^2 + \alpha_2^2, \quad (4.15)$$

$$\tilde{Q}(t) := Q(t)|_{\delta=0} = \prod_{i=1, i \neq j}^{\infty} \left(1 + \frac{\delta_{t-i}}{\bar{d} + i - 1} \right), \quad (4.16)$$

where α_1 is $\sigma\{d_u: u > t - j\}$ -measurable, and α_2 is $\sigma\{d_u: u < t - j\}$ -measurable. Then

$$\begin{aligned} \theta(x, j) &:= A^q(t) Q^p(t) e^{-x^2/2A^2(t)} - \mathbb{E}[A^q(t) Q^p(t) e^{-x^2/2A^2(t)} \mid \mathcal{F}_{t-j}] \\ &= \theta'(x, j) - \theta''(x, j), \end{aligned}$$

where

$$\begin{aligned} \theta'(x, j) &:= A^q(t) Q^p(t) e^{-x^2/2A^2(t)} - \tilde{A}^q(t) \tilde{Q}^p(t) e^{-x^2/2\tilde{A}^2(t)}, \\ \theta''(x, j) &:= \mathbb{E}[A^q(t) Q^p(t) e^{-x^2/2A^2(t)} - \tilde{A}^q(t) \tilde{Q}^p(t) e^{-x^2/2\tilde{A}^2(t)} \mid \mathcal{F}_{t-j}]. \end{aligned}$$

We shall prove the bound

$$|\theta(x, j)| \leq C j^{-1} (1 + x^2) \tilde{A}^q(t) |\tilde{Q}(t)|^p e^{-(x/B)^2/2\tilde{A}^2(t)}. \quad (4.17)$$

It suffices to prove (4.17) for $\theta'(x, j)$ only, as then the corresponding bound for $\theta''(x, j)$ is immediate. Clearly, (4.17) follows from

$$|e^{-x^2/2A^2(t)} - e^{-x^2/2\tilde{A}^2(t)}| \leq C j^{-1} x^2 e^{-(x/B)^2/2\tilde{A}^2(t)}, \quad (4.18)$$

$$|A^q(t) - \tilde{A}^q(t)| \leq Cj^{-1}\tilde{A}^q(t), \quad (4.19)$$

$$|Q^p(t) - \tilde{Q}^p(t)| \leq Cj^{-1}|\tilde{Q}^p(t)|, \quad (4.20)$$

which will be shown below.

According to (3.1), for any $B > 1$ (arbitrary close to 1), we can find $j_0 \geq 0$ such that, for any $j > j_0$,

$$B^{-2} \leq \inf_{|u| \leq D/j} (1+u)^2 \leq \sup_{|u| \leq D/j} (1+u)^2 \leq B^2. \quad (4.21)$$

Clearly, this implies

$$\sup_{|u| \leq D/j} e^{-x^2/2(\alpha_1^2 + (1+u)^2\alpha_2^2)} \leq e^{-(x/B)^2/2\tilde{A}^2(t)} \quad (4.22)$$

and

$$B^{-2}\tilde{A}^2(t) \leq A^2(t) \leq B^2\tilde{A}^2(t), \quad B^{-1}|\tilde{Q}(t)| \leq |Q(t)| \leq B|\tilde{Q}(t)|. \quad (4.23)$$

Let us prove (4.18). We have

$$\begin{aligned} |e^{-x^2/2A^2(t)} - e^{-x^2/2\tilde{A}^2(t)}| &= \left| \exp\left\{-\frac{x^2}{2(\alpha_1^2 + (1+z)^2\alpha_2^2)}\right\} - \exp\left\{-\frac{x^2}{2(\alpha_1^2 + \alpha_2^2)}\right\} \right| \\ &= \left| \int_0^z \left(\exp\left\{-\frac{x^2}{2(\alpha_1^2 + (1+u)^2\alpha_2^2)}\right\} \right)'_u du \right|, \end{aligned}$$

where $z := \delta/(\tilde{d} + j - 1)$. Using the expression of the derivative

$$\left(\exp\left\{-\frac{x^2}{2(\alpha_1^2 + (1+u)^2\alpha_2^2)}\right\} \right)'_u = \exp\left\{-\frac{x^2}{2(\alpha_1^2 + (1+u)^2\alpha_2^2)}\right\} \frac{\alpha_2^2 x^2 (1+u)}{(\alpha_1^2 + (1+u)^2\alpha_2^2)^2}$$

and estimating the right-hand side of the last equation by means of (4.22) and (4.23), relation (4.18) easily follows.

Next, with $z = \delta/(\tilde{d} + j - 1)$,

$$\begin{aligned} A^q(t) - \tilde{A}^q(t) &= (\alpha_1^2 + (1+z)^2\alpha_2^2)^{q/2} - (\alpha_1^2 + \alpha_2^2)^{q/2} \\ &= (q-2)\alpha_2^2 \int_0^z (\alpha_2^2 + (1+u)^2\alpha_2^2)^{(q-2)/2} (1+u) du, \end{aligned}$$

and so (4.19) easily follows from (4.23). Finally, since $Q(t) = \tilde{Q}(t)(1+z)$, (4.20) is immediate from (4.23). This proves (4.17).

Next, from (4.17) and (4.13) we obtain

$$|V(t, j)| \leq C j^{-1} \tilde{A}^q(t) |\tilde{Q}(t)|^p \int_{\mathbb{R}} |h(x)| (1 + |x|^{k+2}) e^{-(x/B)^2/2\tilde{A}^2(t)} dx,$$

and so

$$\begin{aligned} \mathbb{E}U^2(t, j) &\leq \mathbb{E}V^2(t, j) \\ &\leq C j^{-2} \mathbb{E} \left[A^{2q}(t) |\tilde{Q}(t)|^{2p} \left(\int_{\mathbb{R}} |h(x)| (1 + |x|^{k+2}) e^{-(x/B)^2/2A^2(t)} dx \right)^2 \right], \end{aligned}$$

where we used the fact that, for all sufficiently large $j \geq j_0$, one has $\tilde{A}^2(t) \leq A^2(t)$ by (4.14)–(4.15).

Let $1/a + 1/a' = 1$, where $a > 2$ is from (3.7) of Theorem 1. By the Hölder inequality, $I := \int |h(x)| (1 + |x|^{k+2}) e^{-(x/B)^2/2\tilde{A}^2(t)} dx \leq I_1^{1/a} I_2^{1/a'}$, where

$$\begin{aligned} I_1 &:= \frac{1}{\sqrt{2\pi} A(t) B} \int |h(x)|^a e^{-x^2/2A^2(t)B^2} dx = \mathbb{E}[|h(BX)|^a | \mathcal{D}], \\ I_2 &:= (\sqrt{2\pi} A(t) B)^{a'/a} \int (1 + |x|^{k+2})^{a'} e^{-x^2/2A^2(t)B^2} dx, \end{aligned}$$

and $I_2 \leq C A^r(t)$ for suitable $r > 0$ (r can be explicitly found). Therefore (with $\tilde{q} = 2q + 2r/a'$),

$$\begin{aligned} \mathbb{E}[A^{2q}(t) |\tilde{Q}(t)|^{2p} I^2] &\leq \mathbb{E}[A^{\tilde{q}}(t) |\tilde{Q}(t)|^{2p} I_1^{2/a}] \\ &\leq (\mathbb{E}I_1)^{2/a} \left(\mathbb{E}[(A^{\tilde{q}}(t) |\tilde{Q}(t)|^{2p})^{(a/(a-2))}]^{(a-2)/a}, \end{aligned}$$

where the last expectation is finite by Lemma 4.1, and $\mathbb{E}I_1 < \infty$ by condition (3.7). Lemma 4.2 is proved.

LEMMA 4.3. *For any $\epsilon > 0$ and $r > 0$, there exist $N_0 > 0$ and $r_0 > 0$ such that, for all $N > N_0$, the inequalities*

$$|a_j(t)| \leq j^{(\bar{d}+\epsilon)-1}, \quad \forall 1 \leq t \leq N, \quad \forall j > (\log N)^{r_0}, \quad (4.24)$$

$$\sum_{j=0}^{\infty} |a_j(t) a_{j+s-t}(s)| \leq |s-t|^{2(\bar{d}+\epsilon)-1}, \quad \forall 1 \leq t \leq s \leq N, \quad s-t > (\log N)^{r_0} \quad (4.25)$$

hold with probability not less than $1 - Ne^{-(\log N)^r}$.

Proof. Let $\epsilon' := \epsilon/2$, $\epsilon'' := \epsilon'/2$, $\bar{\psi}_j := \psi_j(\bar{d} + \epsilon')$, and $\bar{Q}_j(t) := a_j(t)/\bar{\psi}_j = \prod_{i=1}^j (1 + (\delta_{t-i} - \epsilon')/(\bar{d} + \epsilon' + i - 1))$. In view of the assumption that $|d_t| \leq D$, one can

choose a (nonrandom) $j_0 \geq 1$ large enough so that $|(\delta_{t-i} - \epsilon')/(\bar{d} + \epsilon' + i - 1)| < 3/2$ a.s. for all $i > j_0$ and $t \in \mathbb{Z}$. Then, using the trivial bound $|a_j(t)| \leq D^j$ a.s., one obtains

$$\begin{aligned} |\bar{Q}_j(t)| &= |\bar{Q}_{j_0}(t)| \prod_{i=j_0+1}^j \left(1 + \frac{\delta_{t-i} - \epsilon'}{\bar{d} + \epsilon' + i - 1}\right) \\ &\leq C_{j_0} \exp \left\{ \sum_{i=j_0+1}^j \frac{\delta_{t-i} - \epsilon'}{\bar{d} + \epsilon' + i - 1} \right\}, \end{aligned} \quad (4.26)$$

where $C_{j_0} := D^{j_0}/|\bar{\psi}_{j_0}|$. Introduce the following notation:

$$\begin{aligned} S_{j_0,j} &:= \sum_{i=j_0+1}^j \frac{\delta_{t-i} - \epsilon'}{\bar{d} + \epsilon' + i - 1}, \\ T_{j_0,j} &:= \sum_{i=j_0+1}^j \frac{1}{\bar{d} + \epsilon' + i - 1}, \\ \alpha_{t,j} &:= \sum_{i=1}^j (\delta_{t-i} - \epsilon''). \end{aligned}$$

Thus, $|\bar{Q}_j(t)| \leq C_{j_0} e^{S_{j_0,j}}$. We want to evaluate the probability of the event $\bigcap_{1 \leq t \leq N} \bigcap_{j > K_0} \{S_{j_0,j} \leq 0\}$ or the probability of $\bigcap_{1 \leq t \leq N} \bigcap_{j > K_0} \{|\bar{Q}_j(t)| \leq C_{j_0}\}$. Let $j > K_0$ (for $K_0 \geq 1$ specified below). Then

$$S_{j_0,j} = -\epsilon'' T_{j_0,j} + S_{j_0,K_0} + \sum_{i=K_0+1}^j \frac{\alpha_{t,i} - \alpha_{t,i-1}}{\bar{d} + \epsilon' + i - 1},$$

where the last sum equals $\frac{\alpha_{t,j}}{\bar{d} + \epsilon' + j - 1} - \frac{\alpha_{t,K_0}}{\bar{d} + \epsilon' + K_0} + \sum_{i=K_0+1}^{j-1} \frac{\alpha_{t,i}}{(\bar{d} + \epsilon' + i - 1)(\bar{d} + \epsilon' + i)}$. Therefore, $S_{j_0,j} = S'_{j_0,j} + S''_{j_0,j}$, where

$$\begin{aligned} S'_{j_0,j} &:= -\epsilon'' T_{j_0,j} + S_{j_0,K_0} - \frac{\alpha_{t,K_0}}{\bar{d} + \epsilon' + K_0}, \\ S''_{j_0,j} &:= \frac{\alpha_{t,j}}{\bar{d} + \epsilon' + j - 1} + \sum_{i=K_0+1}^{j-1} \frac{\alpha_{t,i}}{(\bar{d} + \epsilon' + i - 1)(\bar{d} + \epsilon' + i)}. \end{aligned}$$

Let

$$\Omega_{\epsilon, K_0, N} := \bigcap_{1 \leq t \leq N} \bigcap_{j > K_0} \left\{ \left| j^{-1} \sum_{i=1}^j \delta_{t-i} \right| \leq \epsilon'' \right\}. \quad (4.27)$$

By Bernstein's inequality for sums of bounded iid rv's, for any $\epsilon > 0$, one can find $c_0 > 0$ and j_0 such that $\mathbb{P}(|j^{-1} \sum_{i=1}^j \delta_{t-i}| > \epsilon'') \leq e^{-c_0 j}$ holds for all $j \geq j_0$ and, therefore,

$$\mathbb{P}(\Omega_{\epsilon, K_0, N}) > 1 - Ne^{-c_1 K_0} \quad (4.28)$$

holds for all $N > N_0$ and K_0 large enough and some $c_1 > 0$ independent of N, K .

By the definition of $\alpha_{t,j}$, on the set $\Omega_{\epsilon, K_0, N}$, one has $\alpha_{t,j} \leq 0$ (for all $j > K_0$ and $1 \leq t \leq N$) and, therefore, on the same set $\Omega_{\epsilon, K_0, N}$, one has $S''_{j_0, j} \leq 0$ for all $1 \leq t \leq N$. It remains to evaluate $S'_{j_0, j}$ (on the set $\Omega_{\epsilon, K_0, N}$). Clearly, $|S_{j_0, K_0}| \leq C \sum_{i=j_0+1}^{K_0} i^{-1} \leq C \log K_0$ a.s. and $|\frac{\alpha_{t, K_0}}{d + \epsilon + K_0}| \leq C$ a.s. for some (nonrandom) constant C independent of K_0, N . Also, $T_{j_0, j} \geq \int_{j_0}^j x^{-1} dx = \log j - \log j_0$ ($j > j_0$), and we obtain

$$S''_{j_0, j} \leq -\epsilon'' \log j + C \log K_0 + C \leq 0 \quad (4.29)$$

whenever $j \geq (eK_0)^{C/\epsilon''}$. Let $r_0 := 2rC/\epsilon''$ and

$$K_0(N) := \lceil (\log N)^r / c_1 \rceil.$$

Then $j \geq (eK_0(N))^{C/\epsilon''}$ holds for $j > (\log N)^{r_0}$ and, moreover, $c_1 K_0(N) > (\log N)^r$ holds for all $N > N_0$ large enough. We just proved that the inequality

$$\mathbb{P}\left(|a_j(t)| < C_{j_0} |\bar{\psi}_j| \forall 1 \leq t \leq N, \forall j > (\log N)^{r_0}\right) > 1 - Ne^{-(\log N)^r} \quad (4.30)$$

holds for all $N > N_0$ large enough. The statement of the lemma concerning the event (4.24) now follows from the fact that $C_{j_0} \bar{\psi}_j = C_{j_0} |\psi_j(\bar{d} + (\epsilon/2))| < j^{\bar{d} + \epsilon - 1}$ for all $j > j_0(\bar{d}, \epsilon, C_{j_0})$ large enough.

Next, consider the probability of (4.25). By (4.30), for $s - t > (\log N)^{r_0}$ and $N > N_0$, the probability that the following inequalities hold

$$\begin{aligned} & \sum_{j=0}^{\infty} |a_j(t) a_{j+s-t}(s)| \\ & \leq C_{j_0} \sum_{0 \leq j \leq (\log N)^{r_0}} |a_j(t) \bar{\psi}_{j+s-t}| + C_{j_0}^2 \sum_{j > (\log N)^{r_0}} |\bar{\psi}_j \bar{\psi}_{j+s-t}| \end{aligned} \quad (4.31)$$

is not less than $1 - Ne^{-(\log N)^r}$. Using the trivial bound $|a_j(t)| \leq C_2 j^{C_3}$ a.s. for some (nonrandom) $C_2, C_3 > 0$, we see that the right-hand side of (4.31) does not exceed

$$C_4 \left((\log N)^{r_0(1+C_3)} |s - t|^{\bar{d}+\epsilon-1} + |s - t|^{2(\bar{d}+\epsilon)-1} \right) \leq 2C_4 |s - t|^{2(\bar{d}+\epsilon)-1}$$

for some (nonrandom) $C_4 < \infty$ and all $|t - s| > (\log N)^{r'_0}$, $r'_0 := r_0(1 + C_3)/\bar{d}$. To get the final bound as in (4.25), we replace the previous r_0 by r'_0 . Lemma 4.3 is proved.

5. PROOF OF THEOREM 3.1

Without loss of generality, assume that $\mathbb{E}h(X_t) = 0$. Recall the Hermite expansion in (3.4). Accordingly, we write the conditional Hermite expansion

$$h(X_t) = \sum_{k=0}^{\infty} h_k(t), \quad h_k(t) := \frac{1}{k!} g_k(A(t)) H_k(X_t; A(t)) \quad (k \geq 0), \quad (5.1)$$

which converges conditionally in L^2 (i.e., with respect to the conditional probability $\mathbb{P}_{\mathcal{D}}[\cdot] = \mathbb{P}[\cdot|\mathcal{D}]$) a.s. and, therefore, also unconditionally in L^2 for all $t \in \mathbb{Z}$. By the orthogonality property of Hermite polynomials and using the fact that $A^2(t) \geq 1$ a.s., we have

$$\text{var}_{\mathcal{D}}(h(X_t)) = \sum_{k=1}^{\infty} \frac{1}{k!} g_k^2(A(t)) A^2(t) \geq \sum_{k=1}^{\infty} \frac{1}{k!} g_k^2(A(t)), \quad (5.2)$$

$$\text{cov}_{\mathcal{D}}(h(X_t), h(X_{t'})) = \sum_{k=1}^{\infty} \frac{1}{k!} g_k(A(t)) g_k(A(t')) \left(\sum_{j=0}^{\infty} a_j(t) a_{j+t'-t}(t') \right)^k. \quad (5.3)$$

Split $h(X_t) = h'_t + h''_t$, where

$$h'_t := \sum_{0 \leq k \leq k_0} h_k(t), \quad h''_t := \sum_{k > k_0} h_k(t). \quad (5.4)$$

Let us show that there exist (nonrandom) $k_0 \geq 1$ and $r > 0$ such that

$$\mathbb{E} \left(\sum_{t=1}^N h''_t \right)^2 = O(N(\log N)^r). \quad (5.5)$$

In other words, we want to show that partial sums of (h''_t) are negligible with respect to partial sums of (h'_t) which will be shown below to give the limit law of partial sums of $(h(X_t))$ as in Theorem 3.1.

To prove (5.5), split $\mathbb{E}(\sum_{t=1}^N h_t'')^2 = \Sigma_1(N) + 2\Sigma_2(N)$, where

$$\Sigma_1(N) := \sum_{1 \leq t, s \leq N, |s-t| \leq (\log N)^{r_0}} \mathbb{E} h_t'' h_s'', \quad \Sigma_2(N) := \sum_{1 \leq t < s \leq N, s-t > (\log N)^{r_0}} \mathbb{E} h_t'' h_s'',$$

and where $r_0 > 0$ will be determined below. Clearly, $|\Sigma_1(N)| \leq N(\log N)^{r_0} \mathbb{E}(h''(0))^2 \leq N(\log N)^{r_0} \mathbb{E} h^2(X_0)$. Let us prove $\Sigma_2(N) = O(N(\log N)^r)$. Since $k_*^A \geq 1$, $0 < \bar{d} < 1/2$, and $(1 - 2\bar{d})k_*^A < 1$, one can take $\epsilon > 0$ small enough so that $0 < \bar{d} + \epsilon < 1/2$. Choose $k_0 > k_*^A$ such that

$$(1 - 2(\bar{d} + \epsilon))k_0 > 1. \quad (5.6)$$

Let

$$\rho_N(\tau) := \sup_{\substack{1 \leq t < s \leq N \\ s-t=\tau}} \sum_{j=0}^{\infty} |a_j(t) a_{j+s-t}(s)|.$$

By Lemma 4.3, for all $\epsilon > 0$ and $r > 0$, there exist $r_0 > 0$ and $N_0 > 0$ such that, for any $N \geq N_0$,

$$\mathbb{P}(\Omega_{\epsilon, N, r}) := \mathbb{P}(\rho_N(\tau) \leq \tau^{2(\bar{d}+\epsilon)-1}, \forall \tau > (\log N)^{r_0}) \geq 1 - N e^{-(\log N)^r}. \quad (5.7)$$

By the orthogonality property,

$$\begin{aligned} \Sigma_2(N) &= \sum_{s-t > (\log N)^r} \sum_{k > k_0} \frac{1}{k!} \mathbb{E} G_k(s, t), \\ G_k(s, t) &:= g_k(A(s)) g_k(A(t)) \left(\sum_{j=0}^{\infty} a_j(t) a_{j+s-t}(s) \right)^k. \end{aligned}$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned} |G_k(s, t)| &\leq |g_k(A(s)) A^k(t) g_k(A(s)) A^k(t)| \\ &\leq \frac{1}{2} (g_k^2(A(s)) A^{2k}(s) + g_k^2(A(t)) A^{2k}(t)) \end{aligned}$$

and, therefore, by (5.2)

$$\begin{aligned} &\sum_{k > k_0} \mathbb{E} |G_k(s, t)| I(\Omega_{\epsilon, N, r}^c) \\ &\leq \frac{1}{2} \mathbb{E} \left(\sum_{k > k_0} \frac{g_k^2(A(s)) A^{2k}(s)}{k!} + \sum_{k > k_0} \frac{g_k^2(A(t)) A^{2k}(t)}{k!} \right) I(\Omega_{\epsilon, N, r}^c) \end{aligned}$$

$$\leq \frac{1}{2} \mathbb{E}(\text{var}_{\mathcal{D}}(h(X_t)) + \text{var}_{\mathcal{D}}(h(X_t))) I(\Omega_{\epsilon, N, r}^c).$$

Next, by using (5.7), $\mathbb{E}|h(X_0)|^a < \infty$ ($a > 2$), and the Hölder inequality, we have

$$\begin{aligned} \mathbb{E} \text{var}_{\mathcal{D}}(h(X_t)) I(\Omega_{\epsilon, N, r}^c) &\leq \mathbb{E}^{2/a} (\mathbb{E}_{\mathcal{D}} h(X_t))^a (\mathbb{P}(\Omega_{\epsilon, N, r}^c))^{a/(a-2)} \\ &\leq \mathbb{E}^{2/a} |h(X_0)|^a (N e^{-(\log N)^r})^{a/(a-2)}, \end{aligned}$$

implying

$$\sum_{k > k_0} \sum_{1 \leq t, s \leq N} \frac{1}{k!} \mathbb{E} |G_k(s, t)| I(\Omega_{\epsilon, N, r}^c) \leq C N^{2+a/(a-2)} e^{-c_2 (\log N)^r} = O(N)$$

provided that $r > 1 + a/(a-2)$ was chosen large enough; $c_2 := a/(a-2) > 0$.

Finally, by the definition of $\Omega_{\epsilon, N, r}$, for $s - t > (\log N)^{r_0} > 1$, we obtain

$$\begin{aligned} &\sum_{k > k_0} \frac{1}{k!} \mathbb{E} [G_k(s, t) I(\Omega_{\epsilon, N, r})] \\ &\leq \sum_{k > k_0} \frac{1}{k!} \mathbb{E} |g_k(A(s)) g_k(A(t))| \rho_N^k(s - t) I(\Omega_{\epsilon, N, r}) \\ &\leq (|s - t|^{2(\bar{d} + \epsilon) - 1})^{k_0} \sum_{k > k_0} \frac{1}{k!} \mathbb{E} |g_k(A(s)) g_k(A(t))| \\ &\leq C (|s - t|^{2(\bar{d} + \epsilon) - 1})^{k_0} \sum_{k > k_0} \frac{1}{k!} \mathbb{E} (g_k^2(A(s)) + g_k^2(A(t))) \\ &\leq C |s - t|^{-(1 - 2(\bar{d} + \epsilon))k_0} \mathbb{E} h^2(X_0) \\ &\leq C |s - t|^{-(1 - 2(\bar{d} + \epsilon))k_0}, \end{aligned}$$

where we recall that k_0 was chosen so that $(1 - 2(\bar{d} + \epsilon))k_0 > 1$. Hence,

$$\sum_{1 \leq s, t \leq N, s-t > (\log N)^{r_0}} \sum_{k > k_0} \frac{1}{k!} \mathbb{E} [G_k(s, t) I(\Omega_{\epsilon, N, r})] = O(N),$$

thereby proving (5.5).

Let us prove that

$$N^{(1-2\bar{d})(k_*^A/2)-1} \sum_{t=1}^{[N\tau]} h'_t \rightarrow_{\text{fidi}} \frac{\beta_{k_*^A}}{k_*^A!} J_{k_*^A}(\tau), \quad (5.8)$$

$$\mathbb{E}\left(\sum_{t=1}^N h'_t\right)^2 = O(N^{2-(1-2\bar{d})k_*^A}). \quad (5.9)$$

From (5.1), using the properties and notation of Wick products (see Surgailis [12]), we have

$$h'_t = \sum_{k=0}^{k_0} \frac{g_k(A(t))}{k!} \sum a_{j_1}(t) \cdots a_{j_k}(t) : \varepsilon_{t-j_1} \cdots \varepsilon_{t-j_k} : \equiv \sum_{k=0}^{k_0} \frac{1}{k!} Z_k(t), \quad (5.10)$$

where the middle sum is taken over all $j_1, \dots, j_k = 0, 1, \dots$. Next, we decompose each of the “chaotic” terms $Z_k(t)$ as

$$Z_k(t) = Z_{0k}(t) + Z_{1k}(t) + Z_{2k}(t), \quad (5.11)$$

where

$$Z_{0k}(t) := \mathbb{E}[g_k(A(0))Q^k(0)]H_k(Y_t, \Psi), \quad (5.12)$$

$$Z_{1k}(t) := (g_k(A(t))Q^k(t) - \mathbb{E}[g_k(A(t))Q^k(t)])H_k(Y_t, \Psi), \quad (5.13)$$

$$Z_{2k}(t) := g_k(A(t)) \sum (a_{j_1}(t) \cdots a_{j_k}(t) - Q^k(t) \psi_{j_1} \cdots \psi_{j_k}) : \varepsilon_{t-j_1} \cdots \varepsilon_{t-j_k} :, \quad (5.14)$$

where we used the fact that

$$\sum \psi_{j_1} \cdots \psi_{j_k} : \varepsilon_{t-j_1} \cdots \varepsilon_{t-j_k} : = H_k(Y_t, \Psi)$$

is a Hermite polynomial in the Gaussian FARIMA(0, \bar{d} , 0) process $Y_t := \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$ with variance $\Psi^2 := \mathbb{E}Y_0^2 = \sum_{j=0}^{\infty} \psi_j^2$. Note that $Z_{0k}(t) = \beta_k H_k(Y_t, \Psi) \equiv 0$ for $k < k_*^A$, according to the definitions of k_*^A in (3.8) and β_k in (3.6). By (5.10) and (5.11), the proof of (5.8) and (5.9) reduces to

$$N^{(1-2\bar{d})(k/2)-1} \sum_{t=1}^{[N\tau]} H_k(Y_t; \Psi) \rightarrow_{\text{fdd}} J_k(\tau), \quad 1 \leq k < \frac{1}{1-2\bar{d}}, \quad (5.15)$$

$$\mathbb{E}\left(\sum_{t=1}^N H_k(Y_t; \Psi)\right)^2 = O(N^{2-(1-2\bar{d})k}), \quad 1 \leq k < \frac{1}{1-2\bar{d}}, \quad (5.16)$$

$$\mathbb{E}\left(\sum_{t=1}^N H_k(Y_t; \Psi)\right)^2 = O(N \log N), \quad k \geq \frac{1}{1-2\bar{d}}, \quad (5.17)$$

$$\mathbb{E}\left(\sum_{t=1}^N Z_{1k}(t)\right)^2 = O(N \log N), \quad k \geq 0, \quad (5.18)$$

$$\mathbb{E} \left(\sum_{t=1}^N Z_{2k}(t) \right)^2 = O(N \log N), \quad k \geq 0, \quad (5.19)$$

in view of the fact that $k_*^A < 1/(1 - 2\bar{d})$.

Relations (5.15)–(5.17) are well known (see Taqqu [14]). Consider (5.18). Let $\phi_t := g_k(A(t))Q^k(t)$; then, by independence of (d_t) and (ε_t) ,

$$\text{cov}(Z_{1k}(0), Z_{1k}(t)) = \text{cov}(\phi_0, \phi_t) \text{cov}(H_k(Y_0, \Psi), H_k(Y_t, \Psi)).$$

Clearly, the above ϕ_t is a particular case of (4.8) in Lemma 4.2 yielding $\text{cov}(\phi_0, \phi_t) = O(t^{-1} \log t)$, hence, also $\text{cov}(Z_{1k}(0), Z_{1k}(t)) = O(t^{-1} \log t)$ for all $k \geq 0$, thus proving (5.18).

Consider (5.19), which obviously follows from $r_k(t) := \text{cov}(Z_{2k}(0), Z_{2k}(t)) = O(t^{-1} \log t)$. We have

$$\begin{aligned} r_k(t) &= k! \mathbb{E} g_k(A(0)) g_k(A(t)) \sum_{j_1, \dots, j_k \geq 0} (a_{j_1}(0) \cdots a_{j_k}(0) - Q^k(0) \psi_{j_1} \cdots \psi_{j_k}) \\ &\quad \times (a_{t+j_1}(t) \cdots a_{t+j_k}(t) - Q^k(t) \psi_{t+j_1} \cdots \psi_{t+j_k}). \end{aligned}$$

For simplicity, we consider the case $k = 2$. Let $\tilde{Q}_j(t) := Q_j(t) - Q(t)$ so that $a_j(t) = (Q(t) + \tilde{Q}_j(t))\psi_j$. Then $r_2(t)$ can be rewritten as

$$r_2(t) = 2 \sum_{j_1, j_2} \psi_{j_1} \psi_{j_2} \psi_{t+j_1} \psi_{t+j_2} \Lambda_{t, j_1, j_2},$$

where

$$\begin{aligned} |\Lambda_{t, j_1, j_2}| &:= \left| \mathbb{E} \left[g_2(A(0)) g_2(A(t)) (Q(0) \tilde{Q}_{j_1}(0) + Q(0) \tilde{Q}_{j_2}(0) + \tilde{Q}_{j_1}(0) \tilde{Q}_{j_2}(0)) \right. \right. \\ &\quad \left. \left. \times (Q(t) \tilde{Q}_{t+j_1}(t) + Q(t) \tilde{Q}_{t+j_2}(t) + \tilde{Q}_{t+j_1}(t) \tilde{Q}_{t+j_2}(t)) \right] \right| \\ &\leq \|g_2(A(0))\|_b \|g_2(A(t))\|_b \|Q(0) \tilde{Q}_{j_1}(0) + Q(0) \tilde{Q}_{j_2}(0) + \tilde{Q}_{j_1}(0) \tilde{Q}_{j_2}(0)\|_{b'} \\ &\quad \times \|Q(t) \tilde{Q}_{t+j_1}(t) + Q(t) \tilde{Q}_{t+j_2}(t) + \tilde{Q}_{t+j_1}(t) \tilde{Q}_{t+j_2}(t)\|_{b'} \end{aligned}$$

and where $b > 2$, $1/b + 1/b' = 1/2$. Let us check that there exists $b > 2$ such that $\mathbb{E}|g_k(A(0))|^b < \infty$ ($\forall k \geq 1$). Let $2 < b < a$, where $a > 2$ is the same as in the formulation of Theorem 3.1. By the Hölder inequality, $|\mathbb{E}_{\mathcal{D}} h(X) H_k(X, A)|^b \leq \mathbb{E}_{\mathcal{D}}^{b/a} |h(X)|^a \mathbb{E}_{\mathcal{D}}^{b/a'} |H_k(X, A)|^{a'} (1/a + 1/a' = 1)$ and then

$$\mathbb{E} |g_k(A(0))|^b = \mathbb{E} A^{-2kb} |\mathbb{E}_{\mathcal{D}} h(X) H_k(X, A)|^b$$

$$\begin{aligned}
&\leq \mathbb{E} A^{-2kb} \mathbb{E}_{\mathcal{D}}^{b/a} |h(X)|^a \mathbb{E}_{\mathcal{D}}^{b/a'} |H_k(X, A)|^{a'} \\
&\leq \mathbb{E}^{b/a} |h(X)|^a \mathbb{E}^{(a-b)/a} \left(A^{-2kb} \mathbb{E}_{\mathcal{D}}^{b/a'} |H_k(X, A)|^{a'} \right)^{a/(a-b)},
\end{aligned}$$

where the first expectation on the last line is finite because of condition (3.7) of Theorem 3.1 (see also Remark 3.1), and the last expectation is dominated by $\mathbb{E}|A|^q$ for suitable $q < \infty$ and, therefore, is also finite by Lemma 4.1. From Eqs. (4.2) and (4.1), we have that $\|\tilde{Q}_j(t)\|_p \leq Cj^{-1/2}$ and $\|Q(t)\|_p \leq C$ for all $j, p > 1$; hence, the Cauchy–Schwarz inequality yields

$$\begin{aligned}
&\|Q(0)\tilde{Q}_{j_1}(0) + Q(0)\tilde{Q}_{j_2}(0) + \tilde{Q}_{j_1}(0)\tilde{Q}_{j_2}(0)\|_{b'} \leq C(j_1^{-1/2} + j_2^{-1/2}), \\
&\|Q(t)\tilde{Q}_{t+j_1}(t) + Q(t)\tilde{Q}_{t+j_2}(t) + \tilde{Q}_{t+j_1}(t)\tilde{Q}_{t+j_2}(t)\|_{b'} \leq C((t+j_1)^{-1/2} + (t+j_2)^{-1/2}),
\end{aligned}$$

and using arguments as in the proof of relation (4.25) and $\sum_{j>0} |\psi_j \psi_{t+j}| < \infty$, we finally obtain

$$\begin{aligned}
|r_2(t)| &\leq C \sum_{j_1, j_2 \geq 0} |\psi_{j_1} \psi_{j_2} \psi_{t+j_1} \psi_{t+j_2}| (j_1^{-1/2} + j_2^{-1/2}) ((t+j_1)^{-1/2} + (t+j_2)^{-1/2}) \\
&= o(t^{-1}),
\end{aligned}$$

proving (5.19) for $k = 2$.

The case $k \geq 2$ is considered similarly, but now one needs to use the Hölder inequality for products of k factors. Now we have

$$r_k(t) = k! \sum_{j_1, \dots, j_k} \psi_{j_1} \cdots \psi_{j_k} \psi_{t+j_1} \cdots \psi_{t+j_k} \Lambda_{t, j_1, \dots, j_k},$$

where $\Lambda_{t, j_1, \dots, j_k}$ is controlled as above:

$$\Lambda_{t, j_1, \dots, j_k} \leq \|g_2(A(0))\|_b \|g_2(A(t))\|_b \|S(0)\|_{b'} \|S(t)\|_{b'}.$$

The two first factors were already estimated. In the last expression, $S(t)$ is a sum of products of $Q(t)$ and $\tilde{Q}_j(t)$ for $j = t + j_1, \dots, t + j_k$, in which some $\tilde{Q}_j(t)$ appears at least once by using the elementary identity $x^k - y^k = (x - y)(x^{k-1} + \cdots + y^{k-1})$. Exactly the same arguments thus yield this more general result, since $\|S(t)\|_{b'} \leq C\{(t + j_1)^{-1/2} + \cdots + (t + j_k)^{-1/2}\}$. Now we have

$$\begin{aligned}
|r_k(t)| &\leq C \sum_{j_1, \dots, j_k \geq 0} |\psi_{j_1} \cdots \psi_{j_k} \psi_{t+j_1} \cdots \psi_{t+j_k}| \\
&\quad \times (j_1^{-1/2} + \cdots + j_k^{-1/2}) ((t + j_1)^{-1/2} + \cdots + (t + j_k)^{-1/2}) \\
&= o(t^{-1}).
\end{aligned}$$

The proof of the finite-dimensional convergence in Theorem 3.1 (3.9) follows from (5.8) and (5.4), (5.5), (5.9). The tightness in Theorem 3.1 (3.9) follows by the Kolmogorov's criterion, or $\mathbb{E}(\sum_{t=1}^{[N\tau]} X_t)^2 \leq CN^{1+\gamma}$, $\gamma := 1 - (1 - 2\bar{d})k_*^A > 0$, which follows from (5.4), (5.5), (5.9), and the stationarity of (X_t) . This completes the proof of Theorem 3.1.

We end the paper with few examples of nonlinear functions h in which the limit process in Theorem 3.1 (3.9) is identified.

Example 1. Let $h(x) = x^2$. Then $H_2(x; A) = x^2 - A^2$, $g_2(A) = 2$, $g_0(A) = A^2$, $\beta_0 = \mathbb{E}A^2(0)$, $\beta_1 = 0$, and $\beta_2 = 2\mathbb{E}Q^2(0)$. The same coefficients can be obtained by differentiating the function $\mathbb{E}h(X_0 + wQ(0)) = \mathbb{E}(X_0 + wQ(0))^2 = \mathbb{E}A^2(0) + w^2\mathbb{E}Q^2(0)$ (see (3.10)). From Theorem 3.1 we obtain that, for $1/4 < \bar{d} < 1/2$,

$$N^{-2\bar{d}} \sum_{t=1}^{[N\tau]} (X_t^2 - \mathbb{E}X_t^2) \rightarrow_{D[0,1]} \mathbb{E}[Q^2(0)]J_2(\tau),$$

where J_2 is the Rosenblatt process.

Example 2. Let $h(x) = x^3$. Then $H_3(x; A) = x^3 - 3xA^2$, $g_3(A) = 6$, $g_2(A) = 0$, $g_1(A) = 3A^2$, $g_0(A) = 0$, $\beta_0 = 0$, $\beta_1 = 3\mathbb{E}A^2(0)Q(0)$, $\beta_2 = 0$, and $\beta_3 = 6\mathbb{E}Q^3(0)$. From Theorem 3.1 we obtain that, for $0 < \bar{d} < 1/2$,

$$N^{-\bar{d}-(1/2)} \sum_{t=1}^{[N\tau]} X_t^3 \rightarrow_{D[0,1]} 3\mathbb{E}[A^2(0)Q(0)]J_1(\tau),$$

where J_1 is a fractional Brownian motion. Moreover, if $\beta_1 = 3\mathbb{E}A^2(0)Q(0) = 0$ and $1/3 < \bar{d} < 1/2$, then

$$N^{-3\bar{d}+(1/2)} \sum_{t=1}^{[N\tau]} X_t^3 \rightarrow_{D[0,1]} \mathbb{E}[Q^3(0)]J_3(\tau),$$

where J_3 is a Hermite process of order 3.

Example 3. Let $h(x) = I(x \leq y)$ be the indicator function. Then

$$I(x \leq y) = \sum_{k=0}^{\infty} \frac{F^{(k)}(y/A)}{A^k k!} H_k(x; A),$$

where $F(x) = \mathbb{P}(Z \leq x)$, $Z \sim N(0, 1)$, and

$$\beta_k = \frac{1}{\sqrt{2\pi}} \mathbb{E} \left[e^{-y^2/2A(0)} Q^k(0) A^{-k}(0) H_k(y/A(0)) \right] \quad (k \geq 1).$$

In particular, $\beta_1 \equiv \beta_1(y) := (2\pi)^{-1/2} \mathbb{E}[e^{-y^2/2A^2(0)} Q(0)/A^2(0)]$. From Theorem 3.1 we obtain that, for any $0 < \bar{d} < 1/2$,

$$N^{-\bar{d}-(1/2)} \sum_{t=1}^N (I(X_t \leq y) - \mathbb{P}(X_t \leq y)) \rightarrow_{\text{fdd}} \beta_1(y) J_1(1), \quad (5.20)$$

where $J_1(1)$ is a normal random variable. It seems that the convergence in (5.20) can be extended to a functional convergence in the Skorohod space $D(\mathbb{R})$ with the sup-topology, using the argument of Dehling and Taqqu [1]. Note that the limit process in the above equation is degenerate, similarly as in other papers on empirical processes under long memory (see Dehling and Taqqu [1], Ho and Hsing [6], Giraitis and Surgailis [5], Doukhan *et al.* [3], [4], and the references therein).

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P. Doukhan, G. Lang, D. Surgailis. Pavadinimas lietuviškai

Trumpa rexiomė lietuviškai.